Table 1 Comparison of results of numerical integration of Eq. (6) with two-term approximation with internal radiation absent

$\cos\theta$	τ, Ref. 4	τ, Eq. (6)
1.000	1.335	1.352
0.875	1.281	1.293
0.750	1.226	1.223
0.500	1.108	1.002

by collecting the terms according to powers of α, one obtains for τ_1

$$\tau_1 = -(1+\beta)^{-2}\tau_0^{-6}[2\cos\vartheta + 3\sin^2\theta(1+\beta)^{-1}\tau_0^{-4}] \quad (7)$$

Solutions using Eq. (7) are plotted in Figs. 1 and 2. As may be noted from the figures, temperature curves with α as a parameter cross in the region $\pi/3 < \theta < \pi/2$. The temperature, therefore, ceases to be an analytic function of α in that region, which in effect restricts the range of validity of Eq. (6) to about $0 \leq \theta < \pi/3$.

It is interesting to note that all temperature lines seem to cross in the point where $\tau \approx 1$, in the neighborhood of an inflection point. This fact can be used to get information about the temperatures where Eq. (6) breaks down. From the comparison with the data in the Figs. 1 and 2 and those by Nichols,[4] it is seen that the value of the abscissa of the point $\tau = 1$ is close to $\theta = \cos^{-1}\frac{1}{4}$ for the range of α under consideration. Then, using Eq. (6) for $\theta < \pi/3$, together with the information on the approximate location of the point $\tau = 1$, the temperatures over the rest of the shell can be calculated, using as a check the relation implied in Eq. (3).

It is also interesting to discuss the way in which Eq. (6) works in the case of negligible internal radiation ($\beta = 0$), where the deviation of the shell temperature from the no-conduction condition becomes relatively more significant. In Table 1, the data from Ref. 4, p. 28, are shown for comparison with the figures based on $\alpha = 0.25$ and $\beta = 0$.

From Table 1 the conclusion can be drawn that, for $\beta = 0$, Eq. (6) still gives a reasonable approximation, showing to what extent conduction effects alone will reduce the adiabatic wall temperature on a thin spherical shell in space (typical for $0 < \alpha < 0.5$).

Concluding Remarks

The temperatures obtained by the methods discussed in the foregoing are useful in the calculations connected with the output of the spacecraft solar cell powerplants.

However, one more interesting application seems to be possible. As suggested by Reismann and Jurney,[5] the energy flux F for a sphere near the stagnation point may be approximated for hypersonic speed by the equation

$$F = a + b\cos\theta$$

a and b being functions of the freestream Mach number. They may be considered to be constants for conditions where the effects of radiative cooling are of importance. From an inspection of Eq. (1), it is obvious that it is also applicable for this situation, with only minor modifications, if the freestream velocity remains parallel to the ray $\theta = 0$.

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Satellite Dynamics for Small Eccentricity Including Drag and Thrust

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Nomenclature

A_0	= satellite's cross-sectional area
C_D	= drag coefficient
F_r, F_v	= thrust component, radial and normal, respectively
m	= mass of satellite
μ	= gravitational force on unit mass at unit distance
V	= velocity of satellite
ρ	= density of atmosphere
δ	= $\frac{9}{2}(K + \frac{3}{2})^2 - \frac{1}{2}$

THE motion of a satellite in an orbit of small eccentricity was variously considered in the literature, notably by Perkins,[1] Lawden,[2] Newton,[3] Karrenberg et al.,[4] and Parsons.[5] The nonlinear problem with exponential atmosphere has been the subject of outstanding papers by King-Hele,[6] Sterne,[7] and others. In Ref. 1 the linearized problem is treated for constant thrust without consideration of drag; the results are used in Ref. 4 where the drag is treated as a negative (constant) thrust assuming an atmosphere of constant density and assuming that the initial orbit is *exactly* circular. A complete linearized solution, that is, one valid to the first order in $(\Delta r/r)$, including *arbitrary* initial conditions, *variable* density and drag, and an *arbitrary* thrust program is not known to this author. The availability of explicit expressions in closed form is quite useful in preliminary orbit determination (for instance, for a two-point boundary value problem), in rendezvous problems, and in low thrust problems; hence, the necessary calculations were carried out,[8] and the conclusions are presented in the following note.

For a spherical planet with stationary atmosphere, the equations of motion in polar coordinates (r and v) are

$$m(\ddot{r} - r\dot{v}^2 + \mu/r^2) = F_r(t) - \tfrac{1}{2}C_D A_0 \dot{r} V \rho(r) \quad (1)$$

$$m[(r\ddot{v}) + 2\dot{r}\dot{v}] = F_v(t) - \tfrac{1}{2}C_D A_0 r\dot{v} V \rho(r) \quad (2)$$

Let the satellite be observed, at time $t = 0$, to have angular velocity $\dot{v}(0)$ at distance r_0; an angular velocity n_0 [$\neq \dot{v}(0)$] is defined by the well-known relation

$$r_0 n_0^2 = \mu/r_0^2 \quad (3)$$

and the unknown functions $\epsilon(t)$ and $\varphi(t)$ are introduced

$$r(t) = r_0[1 + \epsilon(t)] \quad (4)$$

$$\dot{v}(t) = n_0[1 + \varphi(t)] \quad (5)$$

into the basic equations, which are then linearized by neglecting higher powers of ϵ and φ. For instance, the velocity $V = (\dot{r}^2 + r^2\dot{v}^2)^{1/2}$ becomes $V = r_0 n_0(1 + \epsilon + \varphi)$ to this approximation. Similarly, the linear approximation to the exponential atmosphere $\rho = \rho_0 \exp[(r_0 - r)/H]$ for scale height H becomes

$$\rho(r) = \rho_0(1 - K\epsilon) \quad (6)$$

where $K = r_0/H$ is introduced as a dimensionless quantity. The initial conditions on ϵ and φ at $t = 0$ are

$$\epsilon(0) = 0 \qquad \dot{\epsilon}(0) = An_0 \equiv \dot{r}(0)/r_0 \quad A \ll 1 \quad (7)$$

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$$\varphi(0) = B \equiv (\dot{v}(0)/n) - 1 \qquad B \ll 1 \quad (8)$$

with $r_0, \dot{r}(0)$, and $\dot{v}(0)$ assumed known from observation. The factor n_0 is introduced next to A to render A dimensionless. For the osculating ellipse in terms of the initial conditions, one has[8]

$$a = r_0(1 + B)^2/e^2 \qquad e^2 = A^2(1 + B)^2 + B^2(2 + B)^2 \quad (9)$$

The dimensionless quantity $\alpha = \frac{1}{2}C_D r_0 A_0 \rho_0/m$ is introduced which is the ratio of drag and gravitational force (at $t = 0$) divided by $(1 + 2B)$, to obtain the following equations for the dimensionless radial and angular accelerations:

$$\ddot{\epsilon} + \alpha n_0 \dot{\epsilon} - n_0^2(3\epsilon + 2\varphi) = F_1(t) \quad (10)$$

$$\dot{\varphi} + 2\epsilon + \alpha n_0[2\varphi - \epsilon(K - 2)] = F_2(t) - \alpha n_0 \quad (11)$$

with $F_1(t) = F_r/mr_0$, and $F_2(t) = F_v/mr_0n_0$.

This linear system is solved by a Laplace transformation and leads to the characteristic equation

$$\Delta(s) = s(s^3 + C_1s^2 + C_2s + C_3) = 0$$
$$C_1 = 3\alpha n_0 \qquad C_2 = n_0^2(1 + 2\alpha^2) \qquad C_3 = -2\alpha n_0^3(K + 1) \quad (12)$$

with roots denoted by $s_0 = 0$, $s_1 = \lambda_1$, $s_2 = (\lambda_2 + \lambda_0 j)$, and $s_3 = (\lambda_2 - \lambda_0 j)$. By Cardan's formula the roots are known exactly; for small (αK), it is convenient to have their approximate values which are, to the second-order in α,

$$\lambda_0 = n_0\{1 + \tfrac{1}{2}\alpha^2[9(K + \tfrac{3}{2})^2 - 1]\}$$
$$\lambda_1 = 2\alpha n_0(K + 1) \qquad \lambda_2 = -\alpha n_0(K + \tfrac{5}{2}) \quad (13)$$

In a numerical example for 300 km initial altitude, ballistic parameter = 0.1 m^2/kg, $K = 107.71$, and hence $(\alpha K) \approx 1.2 \times 10^{-3}$. Equations (13) were found to give the roots correctly to five significant figures.

The solution to (10) and (11) for $F_1 = F_2 = 0$(the complementary function) is

$$e(t) = \alpha_0(e^{\lambda_1 t} - 1) + e^{\lambda_2 t}[\alpha_1 \sin\lambda_0 t + \alpha_2(\cos\lambda_0 t - 1)] \quad (14)$$

$$\varphi(t) = B + \beta_0(e^{\lambda_1 t} - 1) + e^{\lambda_2 t}(\beta_1 \sin\lambda_0 t + \beta_2 \cos\lambda_0 t) - \beta_2 \quad (15)$$

where the coefficients α_0 to β_2 are determined from initial conditions and characteristic roots.[8]† With Eq. (13), the expression for $\epsilon(t)$ becomes explicitly

$$\epsilon(t) = [2B - (K + 1)^{-1} + 2A\alpha(K + 2)] \times (e^{2\alpha n_0(K+1)t} - 1) + e^{-\alpha n_0[K+(5/2)]t} \times \{[A + 2\alpha(1 - 2B)(K + 1)] \sin n_0(1 + \delta\alpha^2)t - 2[B + \alpha A(K + 2)][\cos n_0(1 + \delta\alpha^2)t - 1]\} \quad (16)$$

with a similar expression obtained for $\varphi(t)$ from Eq. (15).

It is apparent from Eq. (16) that the initial conditions enter to the same order as the perturbing influence of drag: this is seen by the simultaneous presence of the quantities $2B$ and $(K + 1)^{-1}$ in the first bracket, and A and 2α in the second bracket. Hence, an initial eccentricity even as small as a few thousandths gives rise to terms as large or larger than the drag perturbation, and this circumstance must be carefully considered if the density of the atmosphere is to be estimated by comparison of observed values with mathematical expressions[4] calculated for an initial eccentricity that vanishes *exactly*, that is, for $A = B = 0$.

The initial orbital elements are given in Eq. (9) in terms of A and B; the eccentricity is seen to be of the same order of magnitude as A and B. If the linearized formulas are to be used for an estimate of the influence of eccentricity on some function of time (say, attitude) which is known for a circular orbit, one may set $B = 0$ and $A = e$ in Eqs. (14) and (15) to obtain explicit expressions in terms of e.

† Mathematical details, being simple exercises, were deleted at Editor's request.

Some improvements in the accuracy of the linearized solutions can be achieved by the following considerations:

1) The motion in the initial osculating orbit is approximated in the linearized solutions by the terms $[A \sin n_0 t - 2B(\cos n_0 t - 1)]$ and $[-2A \sin n_0 t + 4B \cos n_0 t - 3B]$, respectively, obtained by setting $\alpha = 0$ in (14) and (15). The *deviation* from *these* terms constitutes the linearized perturbation due to drag and/or thrust. Therefore, the following expressions are formed:

$$E(t) = \epsilon(t) - A \sin n_0 t + 2B(\cos n_0 t - 1) \quad (17)$$

$$\Phi(t) = \varphi(t) - 4B \cos n_0 t + 2A \sin n_0 t + 3B \quad (18)$$

which represent the linearized perturbations of the initial osculating orbit. By superimposing Eqs. (17) and (18) onto the Kepler ellipse calculated for initial values A and B in the standard way, the inaccuracies incurred by the process of linearization are greatly reduced.

The error in r and $\dot{v}$ is of the order of the first neglected terms, that is, of the order $(r_0\epsilon)^2$ and $(n_0\varphi)^2$, respectively; of the two errors, the in-track error (in v) is the larger. As a numerical example, the order of magnitude of Φ was calculated for the example just cited (300-km alt). For a time interval of 4 orbits ($\lambda_0 t \approx 8\pi$) and eccentricity $e = 0.01$, the resulting error is found to be about 60 m. Considering the simplicity of formulas employed, this is a rather small error.

2) Since the density varies strongly with position, the validity of the formulas is essentially limited by the inaccuracy inherent in the linearization of the exponential variation of density, namely Eq. (6). If the initial excursions $(r_0\epsilon)$ are of the order of a scale height, it is advantageous to use a linear formula for $\rho(r)$ that is more accurate for larger values of the density at perigee than for the small, and therefore less important values at apogee. Let ϵ_1 and ϵ_2 be the initial maximum and minimum of ϵ; then

$$\epsilon_{1\,2} = 2B\{1 \pm [1 + (A^2/4B^2)]^{1/2}\} \quad (19)$$

as follows from Eq. (9) with sufficient accuracy for the present purposes. With ρ_1 and ρ_2, the approximate densities at apogee and perigee, respectively, ρ_0 is replaced in Eq. (6) by

$$\rho_x = \frac{\rho_2}{1 + (|\epsilon_2|/\epsilon_1)} \quad (20)$$

and Eq. (6) is replaced by the following expression:

$$\rho(r) = \rho_x[1 - (\epsilon/\epsilon_1)] \quad (6')$$

Equation (20) yields a slightly different value for α, and, in Eq. (6′), K is set equal to $(1/\epsilon_1)$; this results in a linear interpolation for the density, which is quite accurate near perigee but which neglects the density near apogee. Since the drag decreases by an order of magnitude over the distance of two scale heights, Eq. (6′) will yield better accuracy than Eq. (6) if the initial values A and B are about as large as $(1/K)$.

3) Inasmuch as terms of order ϵ^2 and φ^2 were neglected in the linearized solution, the error, as mentioned before, is of the order of these quantities. For small thrust and/or density, ϵ and φ are proportional to the time interval t, and, hence, the error increases as t^2. However, since the calculations using Eqs. (14, 15, or 16) are very simple, one can subdivide the interval of interest, say T, into n subintervals t_n and repeat the calculation n times using the result obtained for t_{i-1} as the initial condition for the subsequent calculation for the interval t_i. By this procedure the magnitude of the error is reduced by a factor n. For instance, in the forementioned numerical example, the error for 4 orbits was found to be about 60 m; if, instead, the calculations were performed consecutively 4 times, once for each orbit, the cumulative error would be only one-fourth or about 15 m.

By the form of Eqs. (14) and (15), a particularly easy way can be chosen for such calculations: if consecutive intervals

t_n are chosen so that $t_n = 2n\pi/\lambda_0$ (n = any integer), the sine-cosine terms are seen to vanish, and only the first exponential term and constants remain; the radial displacements ϵ_n at time t_n are then given by the particularly simple relation

$$\epsilon_n = \alpha_0(e^{2n\,\pi\lambda_1/\lambda_0} - 1) \tag{21}$$

and a similar expression results for φ_n. For any conveniently chosen integer n, ϵ_n and φ_n constitute new initial conditions for the next time interval. The accuracy of the final results obtained in this manner is *better than first order* in ϵ and φ; though convergence has not been proved, the assumption seems reasonable that by further subdivision of time intervals Eqs. (14) and (15) yield a sequence of solutions which tends to the exact solution of the problem. For *low thrust* devices that often necessitate integrations over hundreds of orbits, the method just outlined will have computational advantages.

The *inclusion of thrust* in the forementioned formulism involves merely the addition of terms, since the basic equations are linear (cf., the previous footnote). If the Laplace transforms of $\epsilon(t)$ and $\varphi(t)$ are known and if $\bar{F}_1(s)$ and $\bar{F}_2(s)$ are the transforms of thrust-functions $F_1(t)$ and $F_2(t)$, then, to include the thrust, $[A + \bar{F}_1(s)]$ must be inserted in place of A and $[B + \bar{F}_2(s)]$ in place of B, and the inverse transforms then yield the complete response that is the complementary function and a particular integral.

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Comparison of Error Transfer Matrices for Circular Orbits

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In problems dealing with the motion of orbiting bodies, it is often necessary to investigate the propagation of errors in position and velocity as the body progresses in its orbit. A convenient tool for such studies is the error matrix, which relates position and velocity errors at two arbitrary points of an orbit.

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For circular orbits, there are two matrices in common use. The first, which may be referred to as the Clohessy-Wiltshire[1] matrix, refers to the relative motion between an orbiting body and a reference satellite in circular orbit. The relative vector is expressed in terms of a Cartesian coordinate system centered on the reference satellite and rotating at a constant angular rate in the reference orbital plane to keep one axis always horizontal. The second formulation may be called the Duke[2] matrix because of its use in referenced reports. This also describes the relative motion between an orbiting body and a reference satellite in circular orbit, but a different coordinate system is employed. In this case the quantities employed are the relative differences in distance to the center of the force field, central angle subtended, velocity vector magnitude, and velocity vector angle with respect to the vertical. A recent paper by Wisneski[3] presents a detailed approach to the derivation of the Duke matrix.

Because of the difference in coordinate systems employed, the two formulations appear to be different. Nevertheless, they are in fact exactly equivalent, and each is based upon the same linearizing assumptions and approximations. It is the purpose of this note to show the equivalence of the two formulations. To avoid confusion, the first formulation will be henceforth referred to as Clohessy-Wiltshire, and the second will be called Duke.

Derivation of Transformation Equations

The pertinent geometry is given in Fig. 1. The reference satellite in circular orbit is labeled S; the orbiting body of interest is located at P. A planar situation only is described. Although a third dimension (perpendicular to the orbital plane) and a time perturbation may be added, these have been omitted since the matrix description of each formulation gives identical terms for each quantity.

The satellite is moving counterclockwise at constant translational velocity V, angular velocity ω, and distance R from the center of the force field O. In the Clohessy-Wiltshire system the relative position of the orbiting body is measured with respect to the rotating Cartesian axes labeled x-y. The coordinates are designated Δx, Δy for position, and $\Delta\dot{x}$, $\Delta\dot{y}$ for velocity.

For the Duke system, the reference satellite is described by the parameters R, θ, V, and β. The orbiting body is described by R_p, θ_p, V_p, and β_p. The relative parameters are then

$$\Delta R = R_p - R \qquad \Delta\theta = \theta_p - \theta$$
$$\Delta V = V_p - V \qquad \Delta\beta = \beta_p - \beta$$

Note that β is actually constant at 90° because of the circularity of the reference satellite's orbit.

The relations between the Clohessy-Wiltshire and Duke systems are derived as follows, noting that

$$\Delta R \ll R \qquad \Delta\theta = \text{very small angle}$$
$$\Delta V \ll V \qquad \Delta\beta = \text{very small angle}$$

For horizontal position,

$$\Delta x = -(R + \Delta R)\sin\Delta\theta \cong -R\Delta\theta \tag{1}$$

For vertical position,

$$\Delta y = (R + \Delta R)\cos\Delta\theta - R \cong \Delta R \tag{2}$$

For horizontal velocity,

$$\Delta\dot{x} = -V_p\sin(\beta_p + \Delta\theta) + \omega\Delta y + V$$

Here the $\omega\Delta y$ term arises because of the rotation of the coordinate system.